Block Trees

[Belazzougui et al., DCC 2015]
[Belazzougui et al., J. CSS, 2021]

Block Tree: Example

a b a b b a a b b b a a b b b a a b b b a b a b b a a b b b a a

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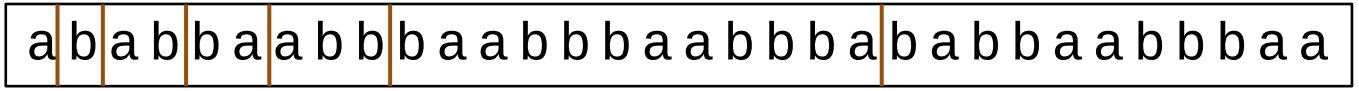

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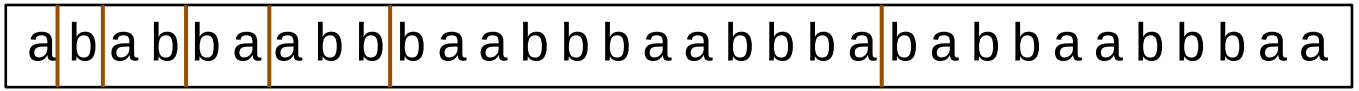

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root degree s := 8

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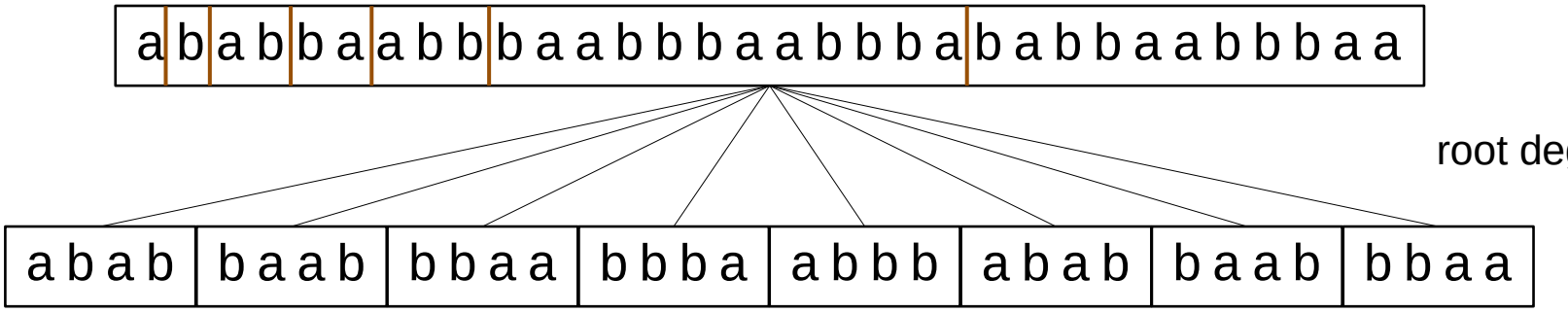

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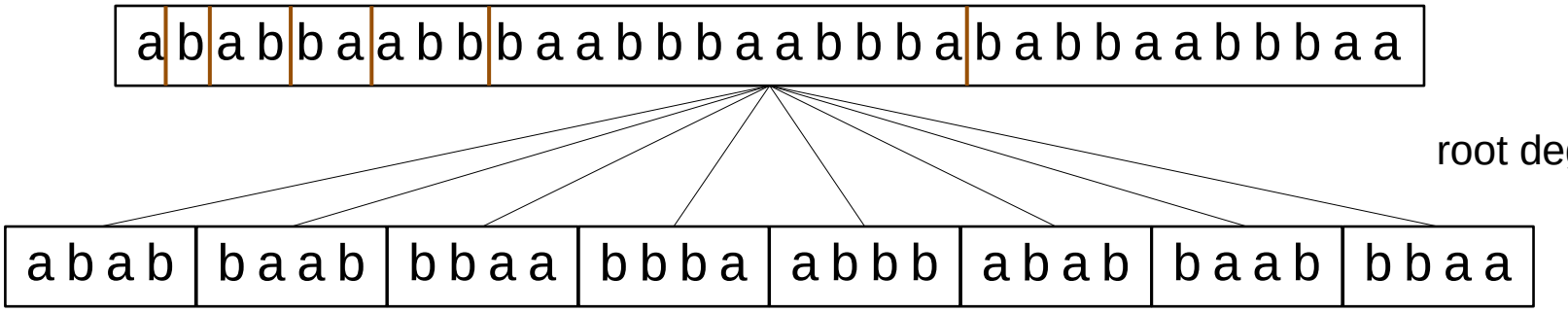

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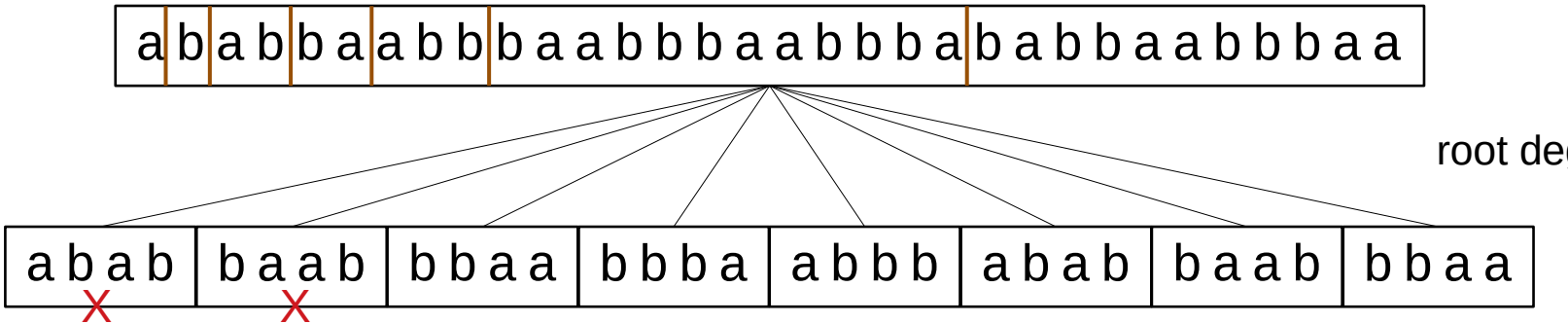

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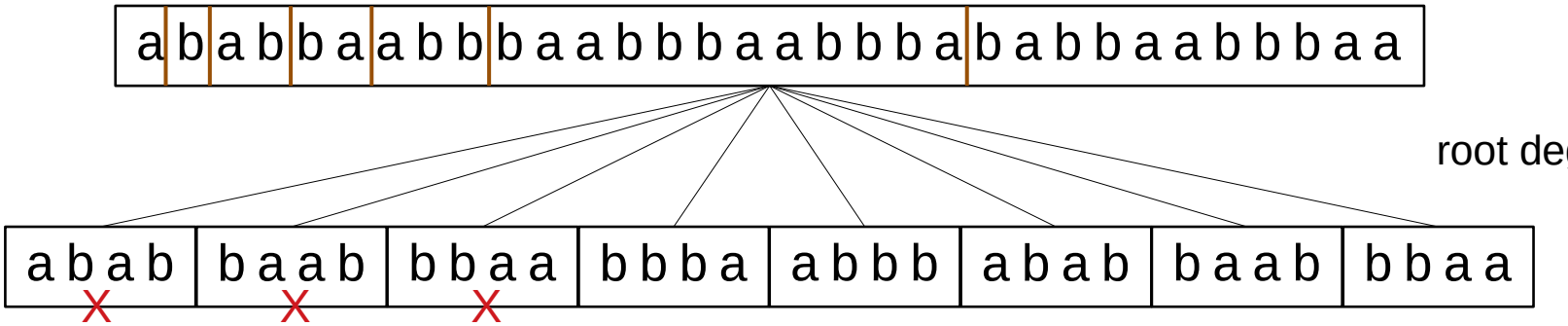

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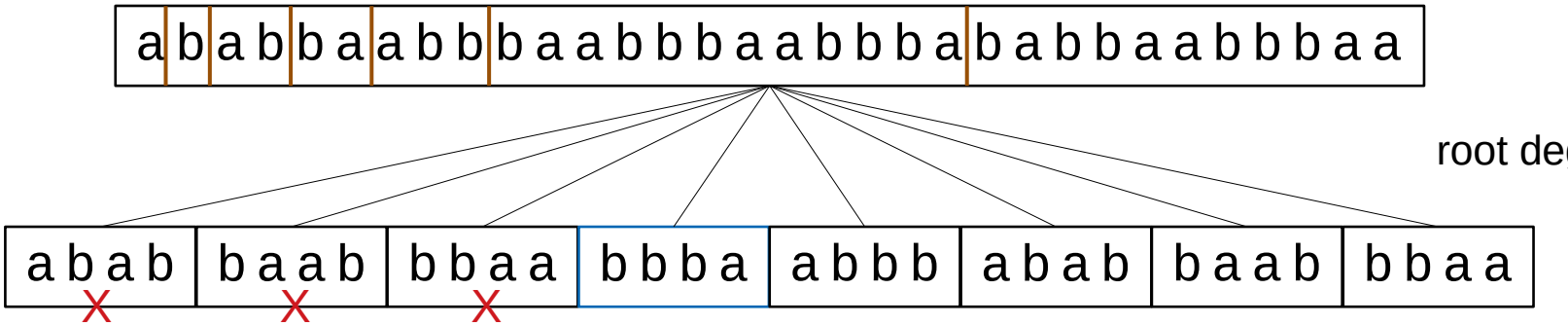

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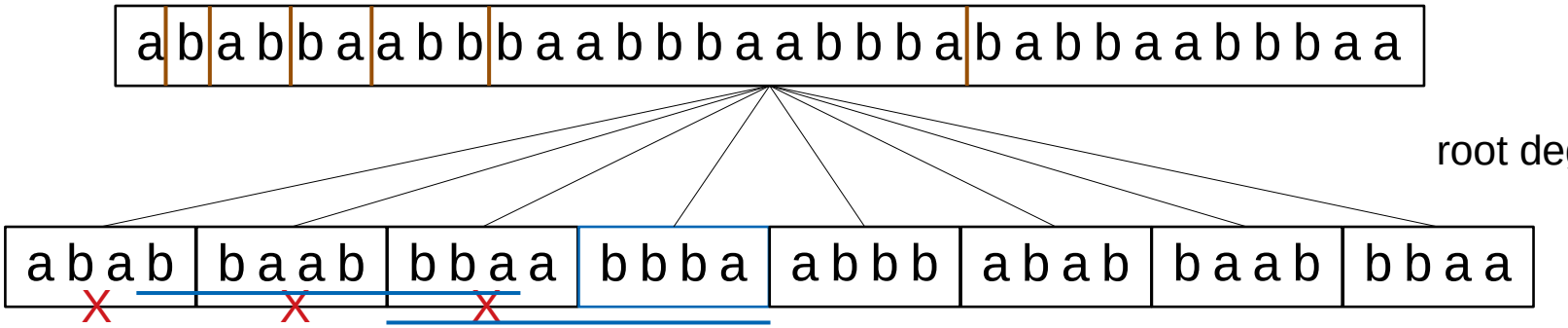

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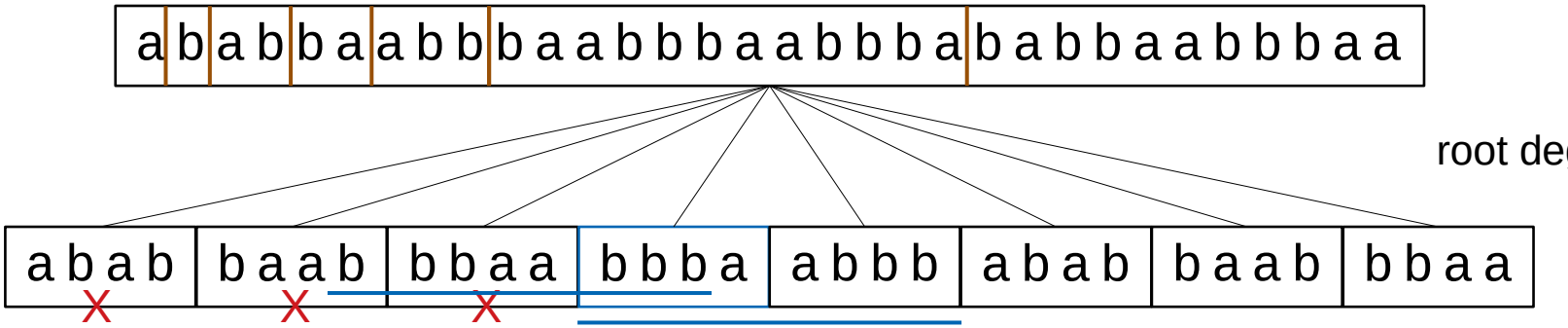

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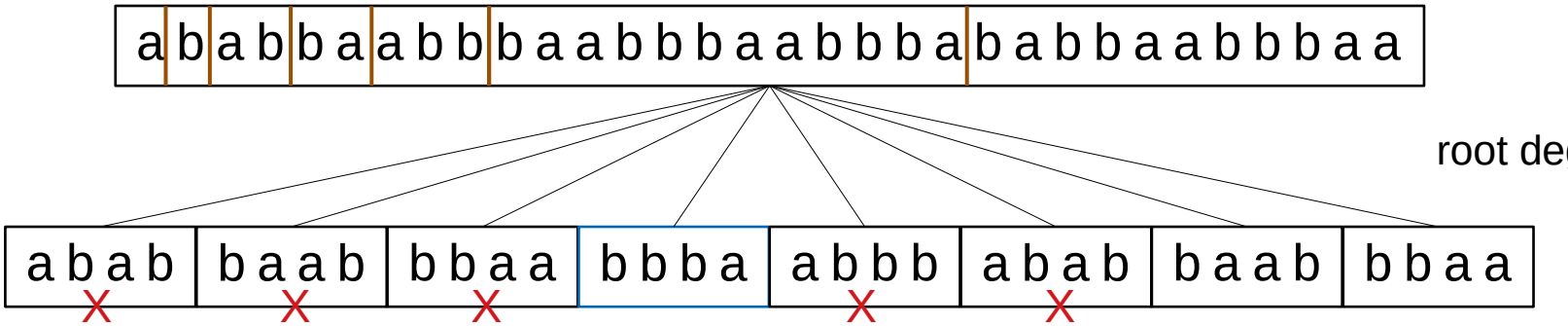

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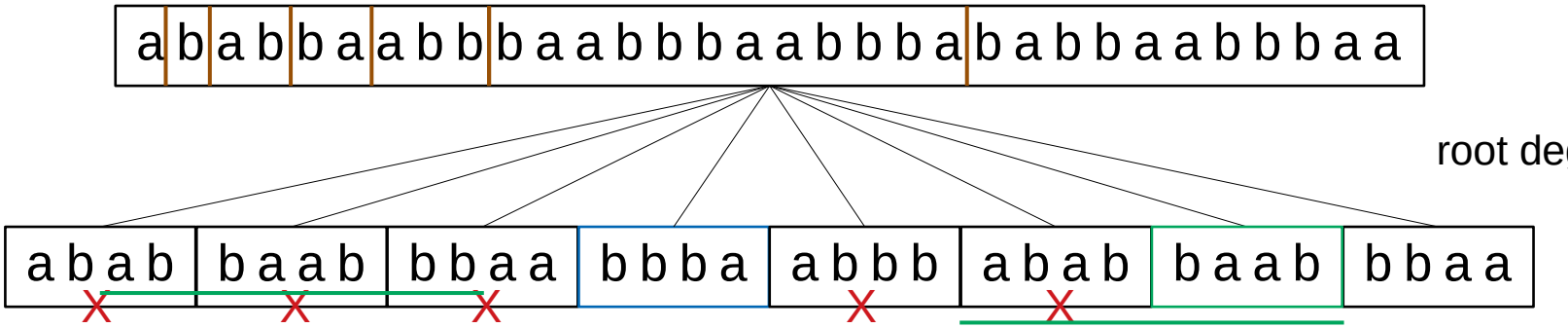

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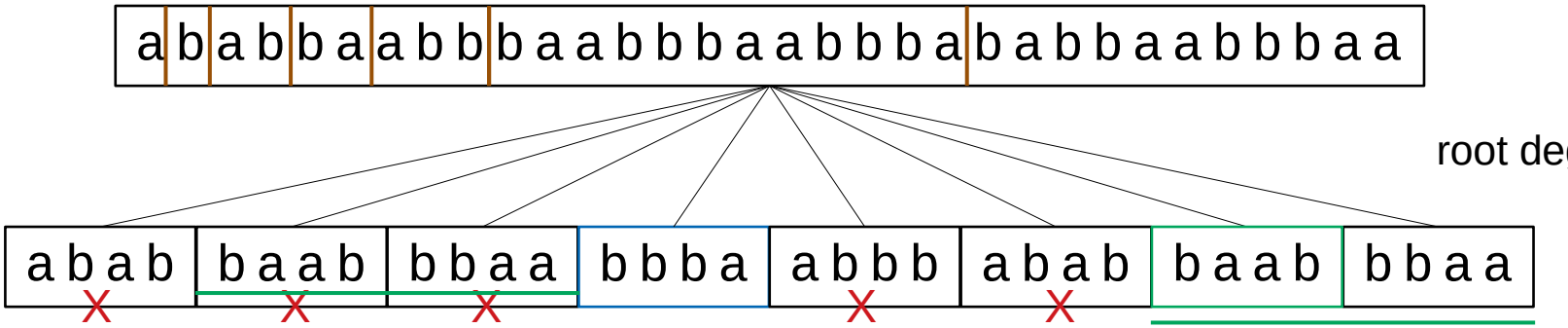

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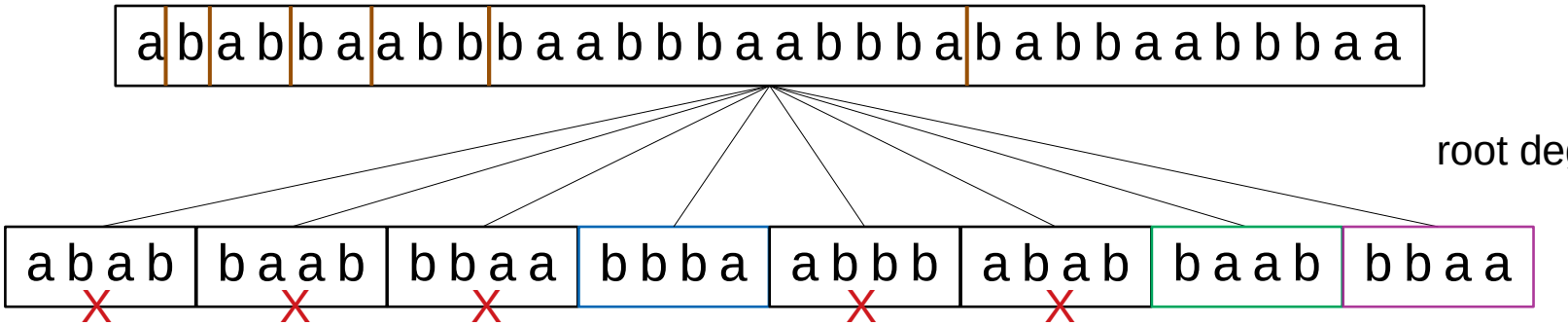

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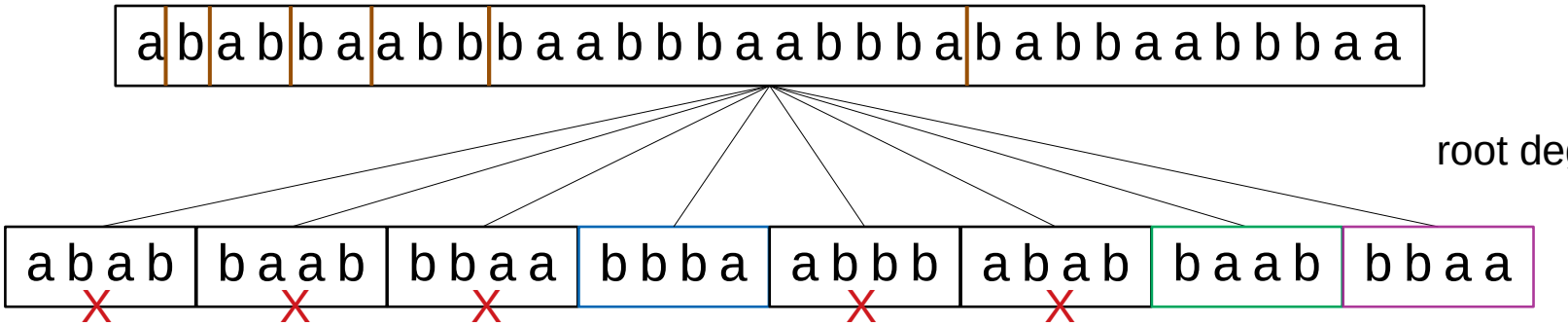

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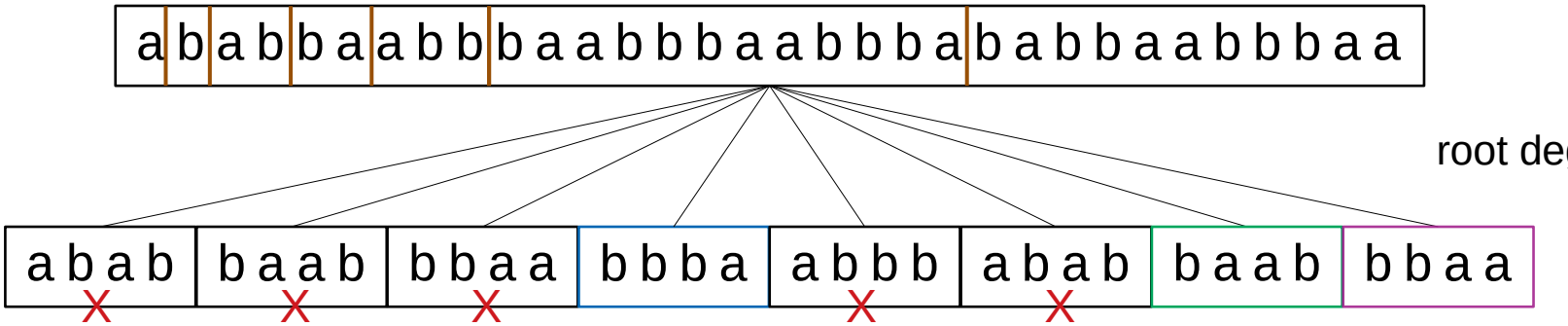

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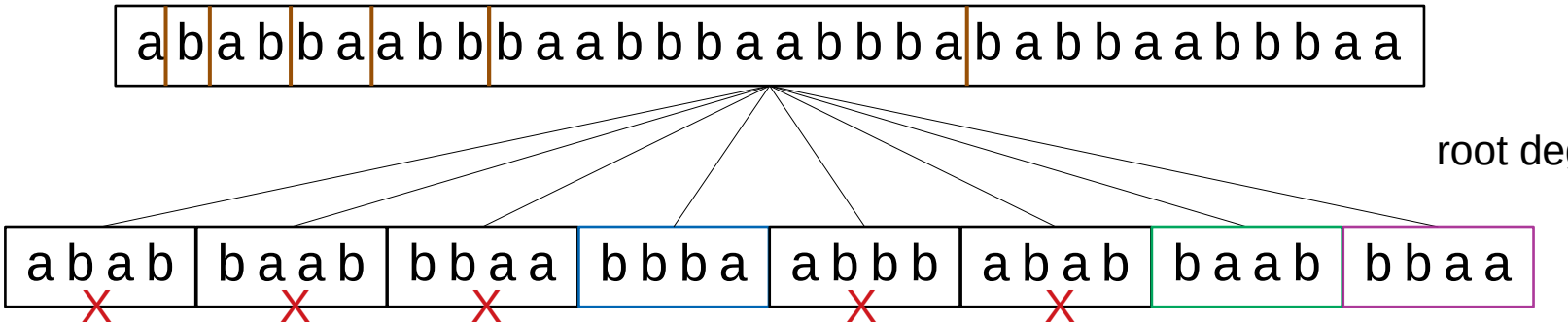

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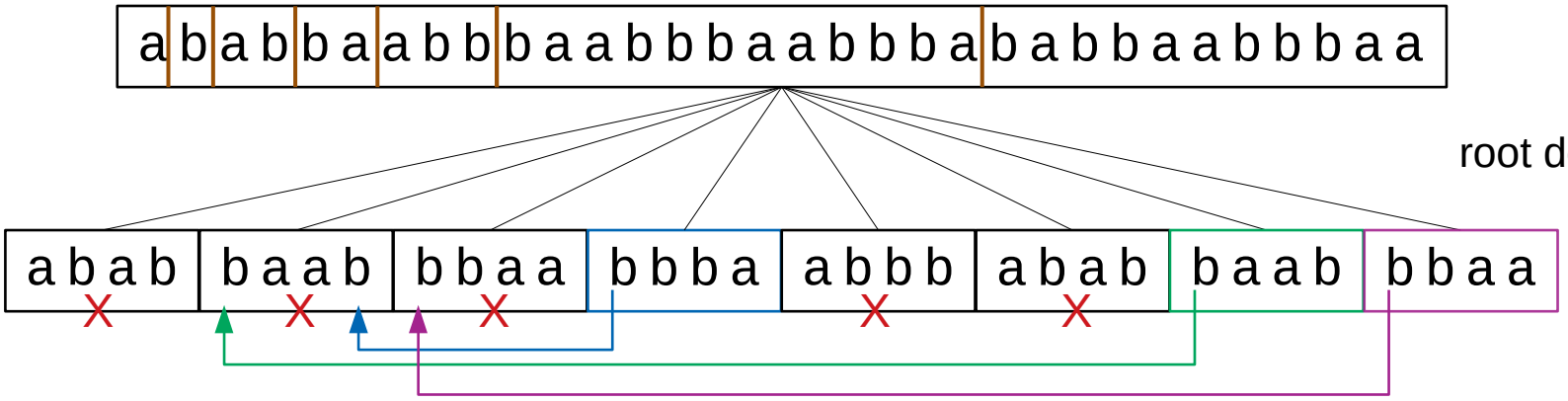

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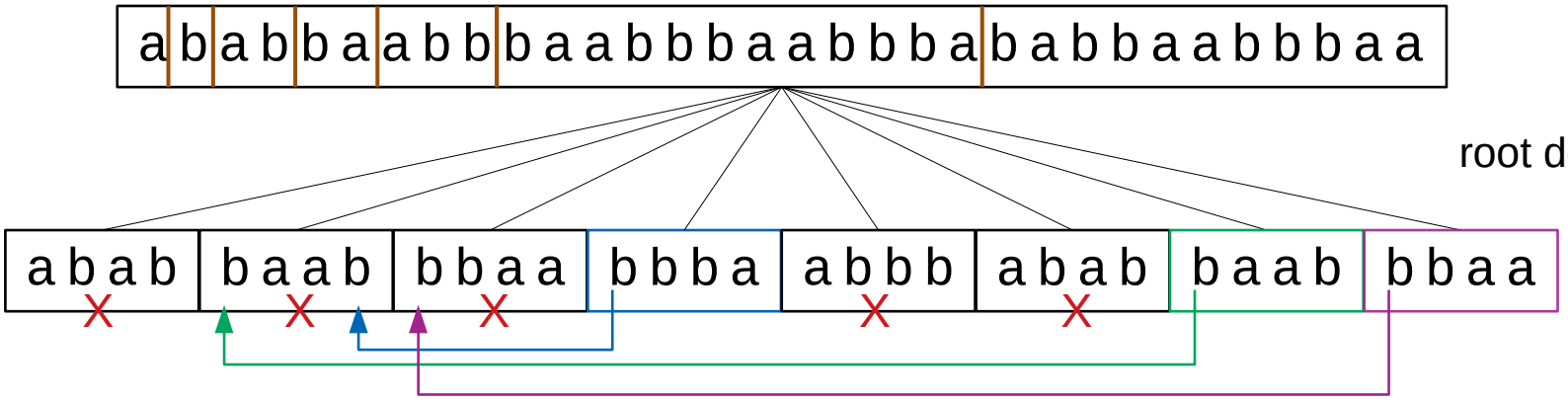

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- For marked blocks, recurse!

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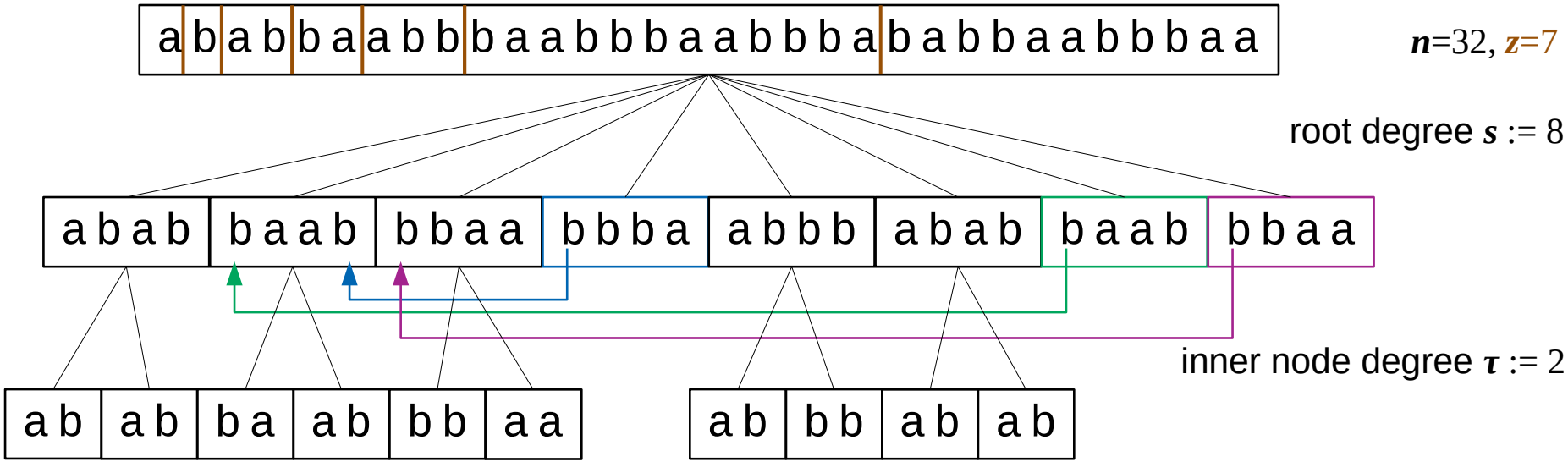

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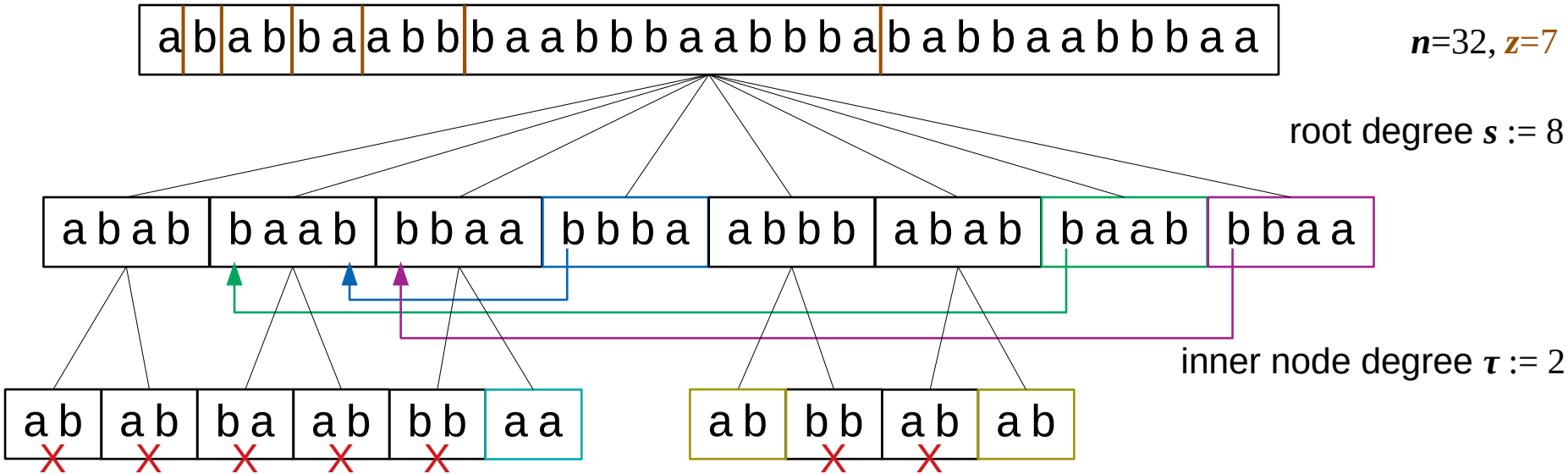

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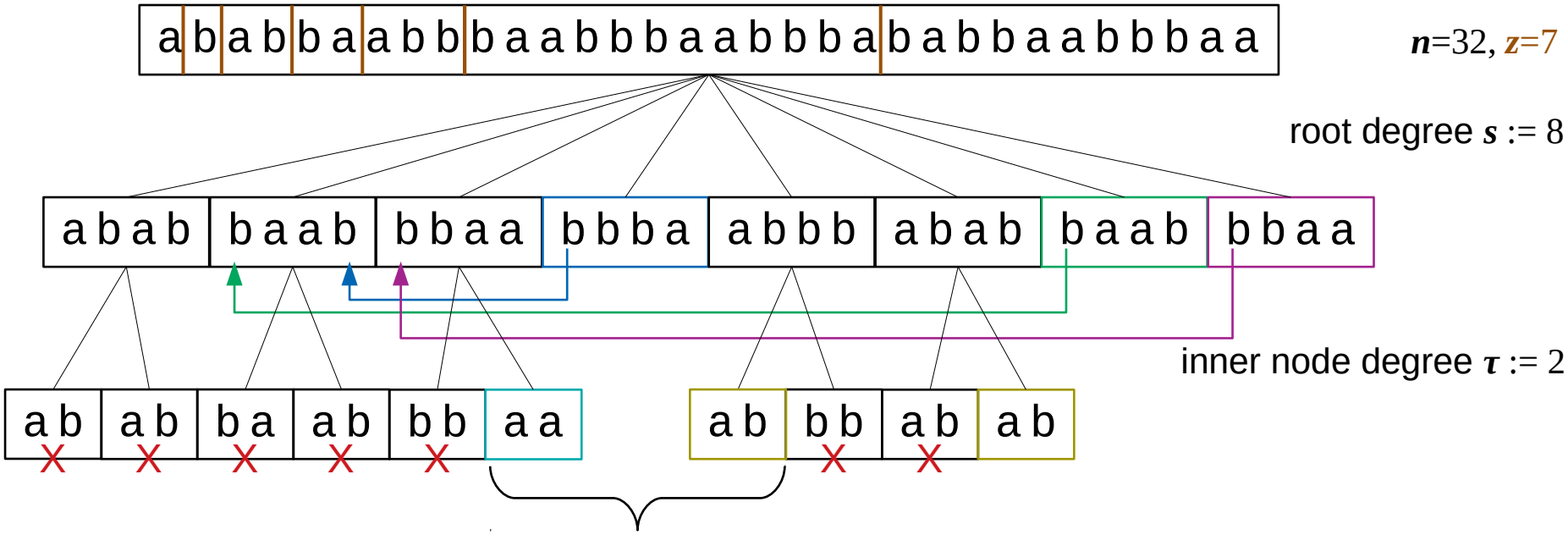

Letfmost occurrence of aaab, but not consecutive in S!

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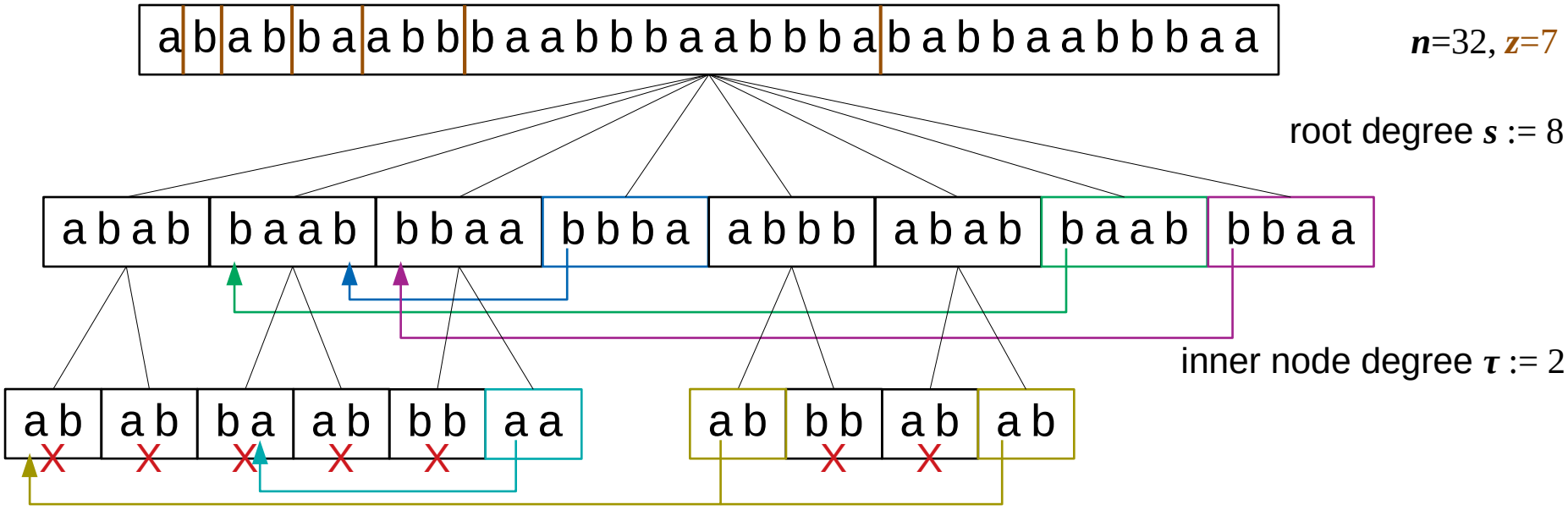

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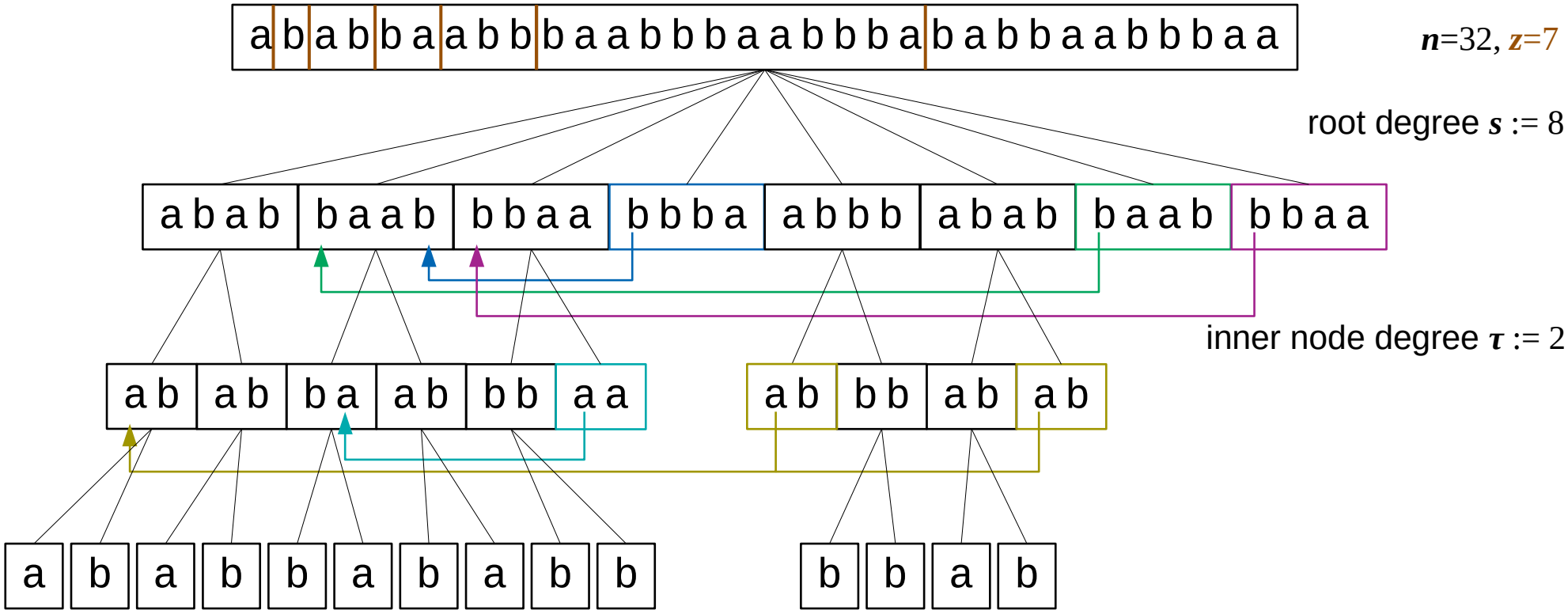

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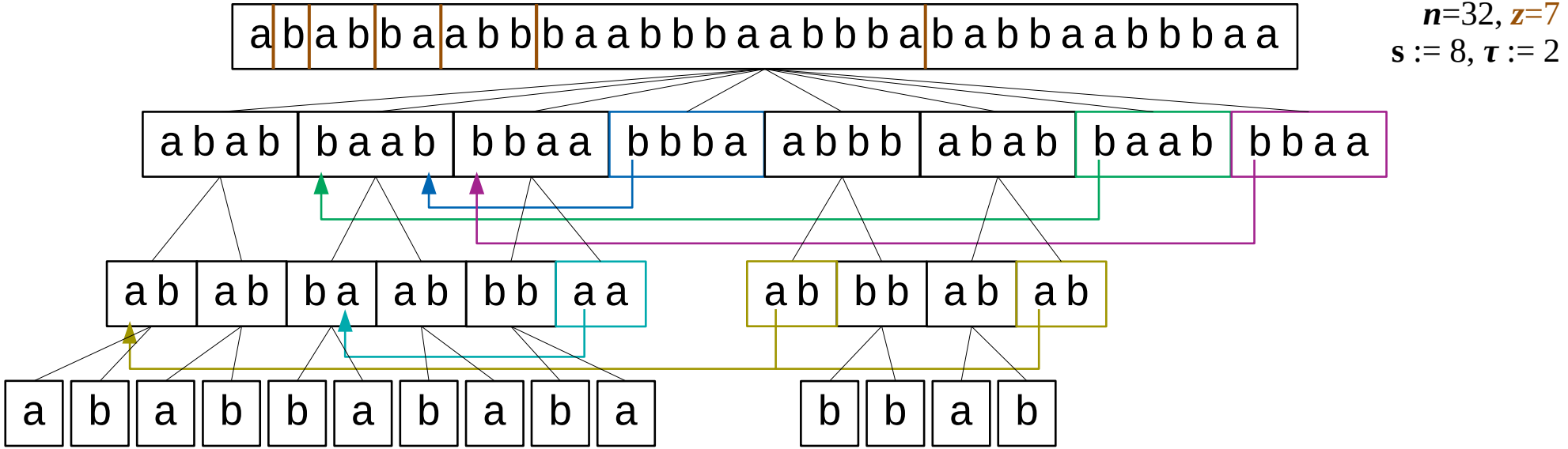

n=32, z=7
$s := 8$, $\tau := 2$

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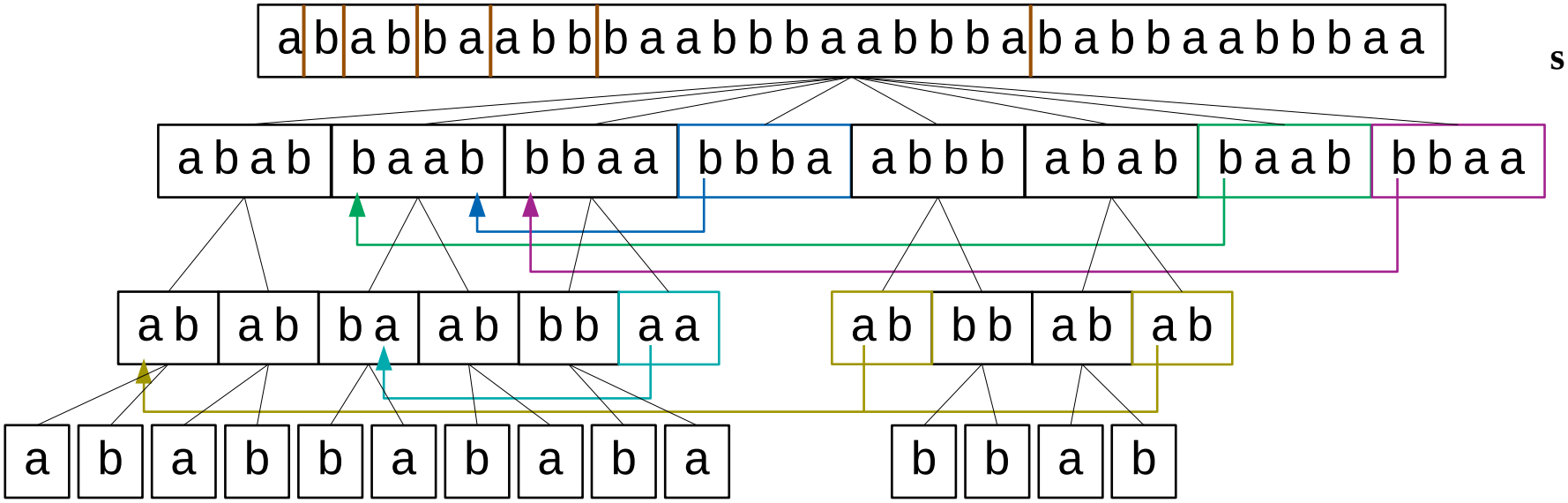

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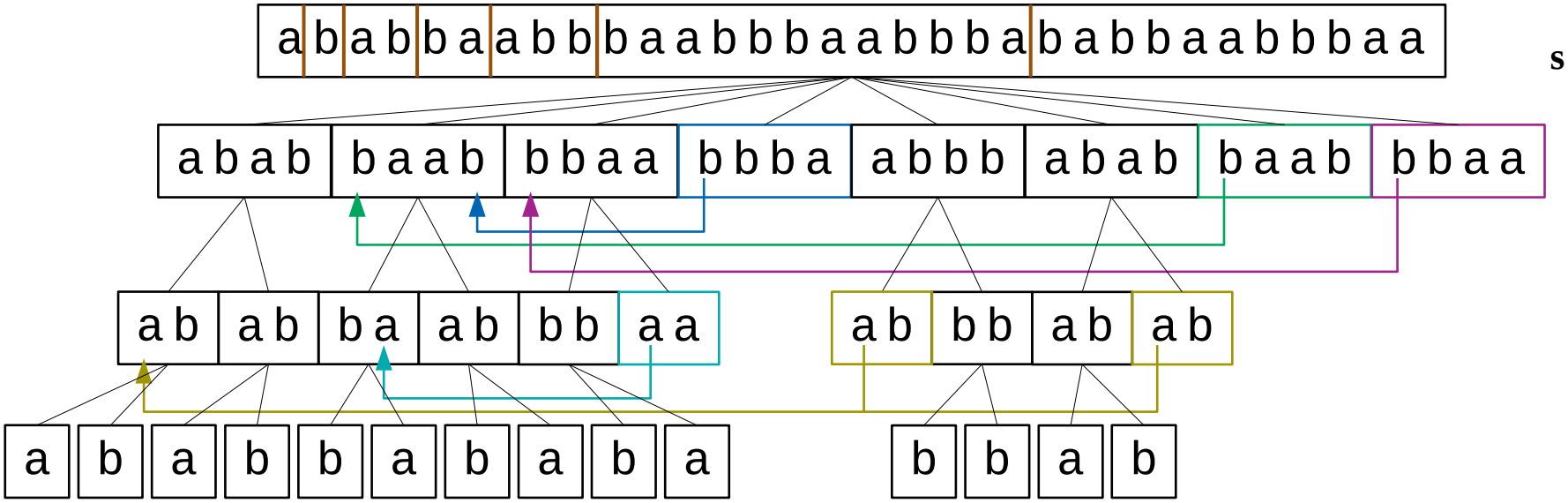

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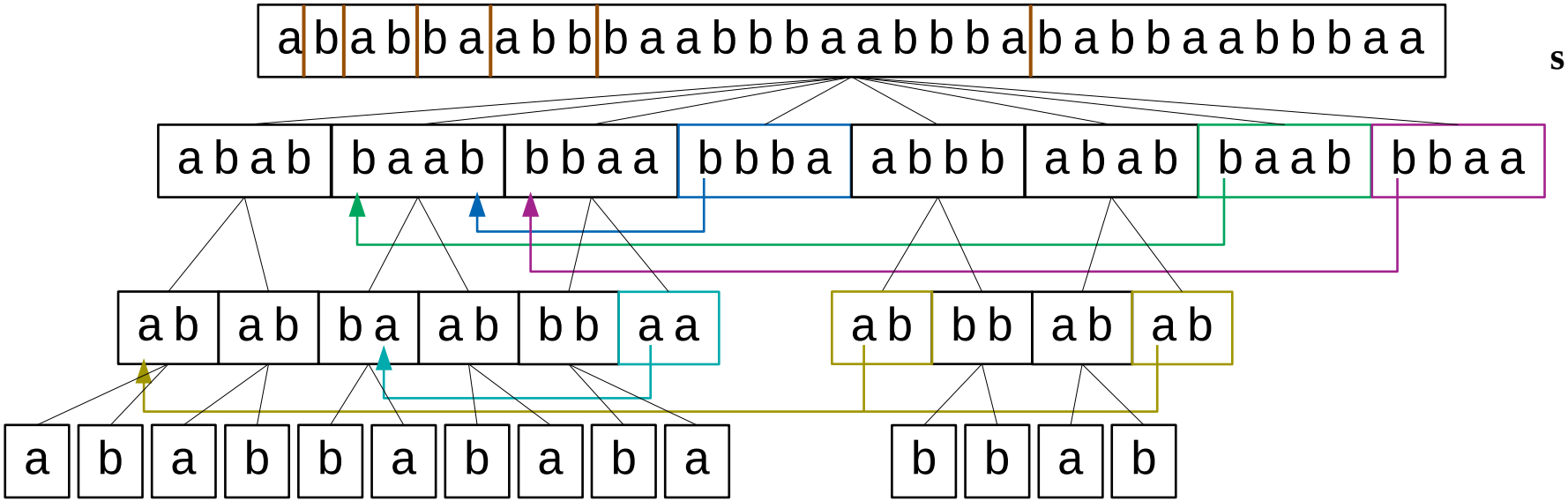

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- The height of the tree is then:

$$h = \log_\tau \left(\frac{n}{s}\right) - \log_\tau(\log_\sigma n) = \log_\tau \frac{n \log \sigma}{s \log n}$$

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  - An unmarked block induces τ blocks on the next level □

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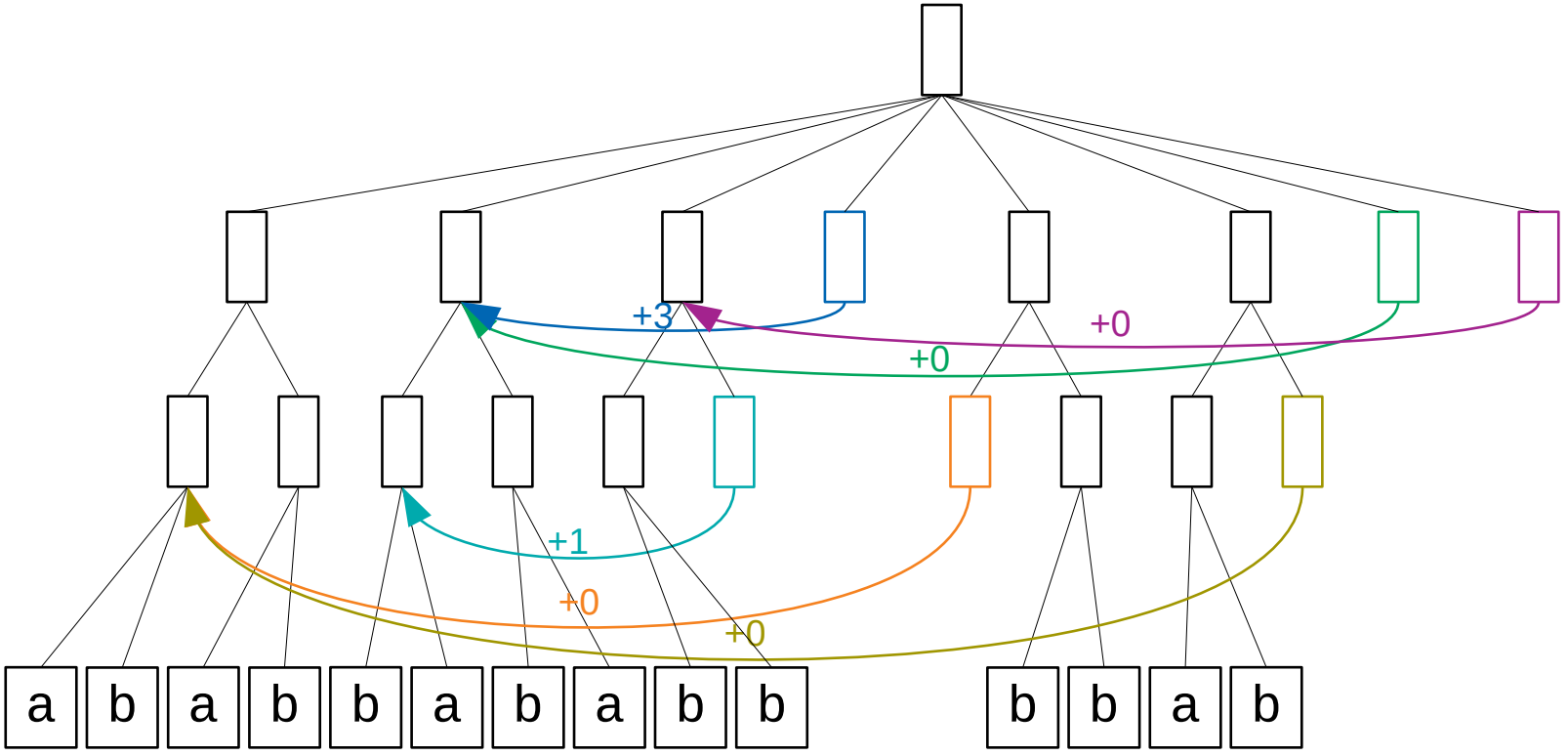
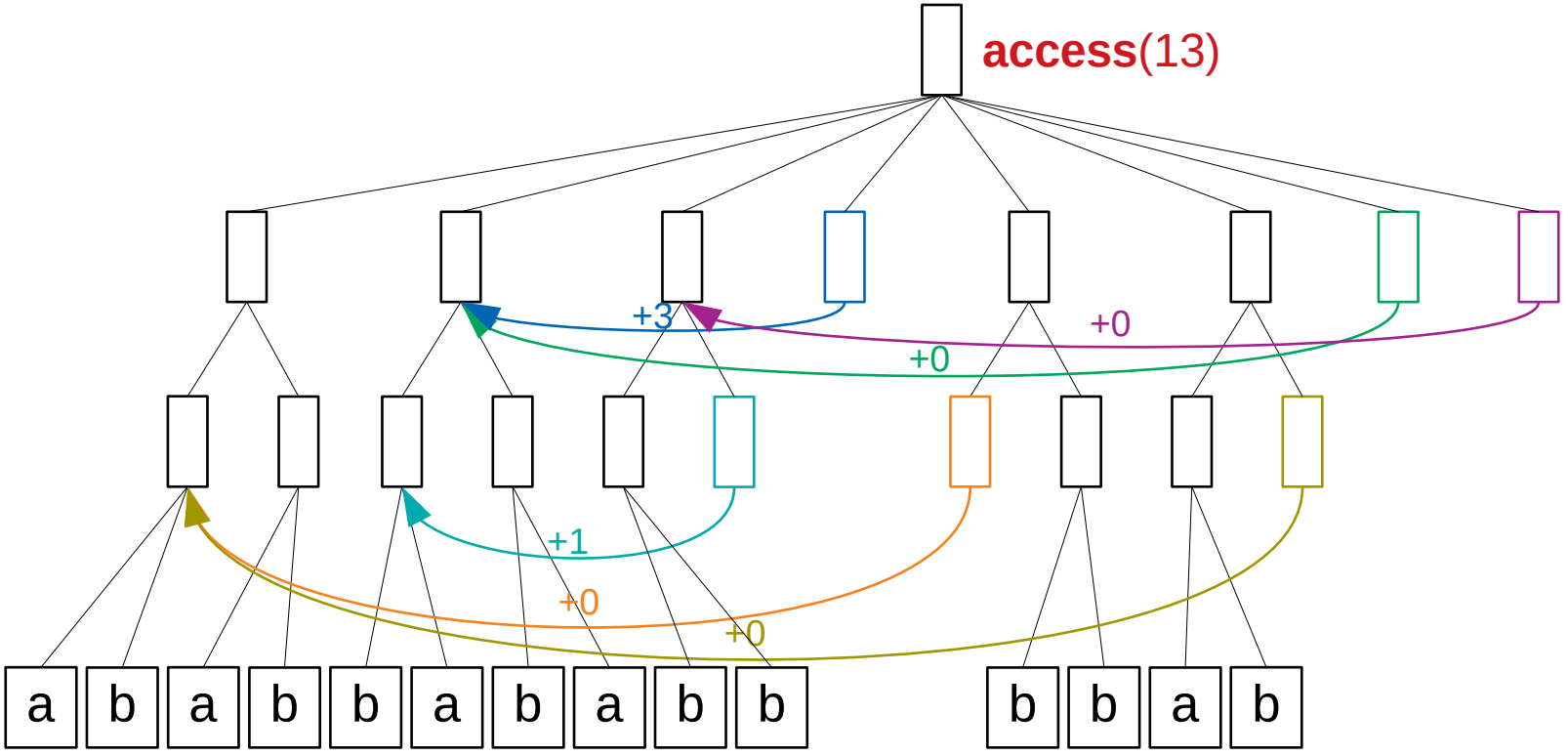
- We store unmarked blocks in $O(\lg n)$ bits (pointers)
- Leaves are stored as plain text in $\Theta(\log_\sigma n) \lg(\sigma) = O(\lg n)$ bits
- The total space required to store the block tree is thus (in bits):

$$\mathcal{O}\left(\left(s + z\tau \log_\tau \frac{n \lg \sigma}{s \lg n}\right) \lg n\right)$$

Storage



Access Query



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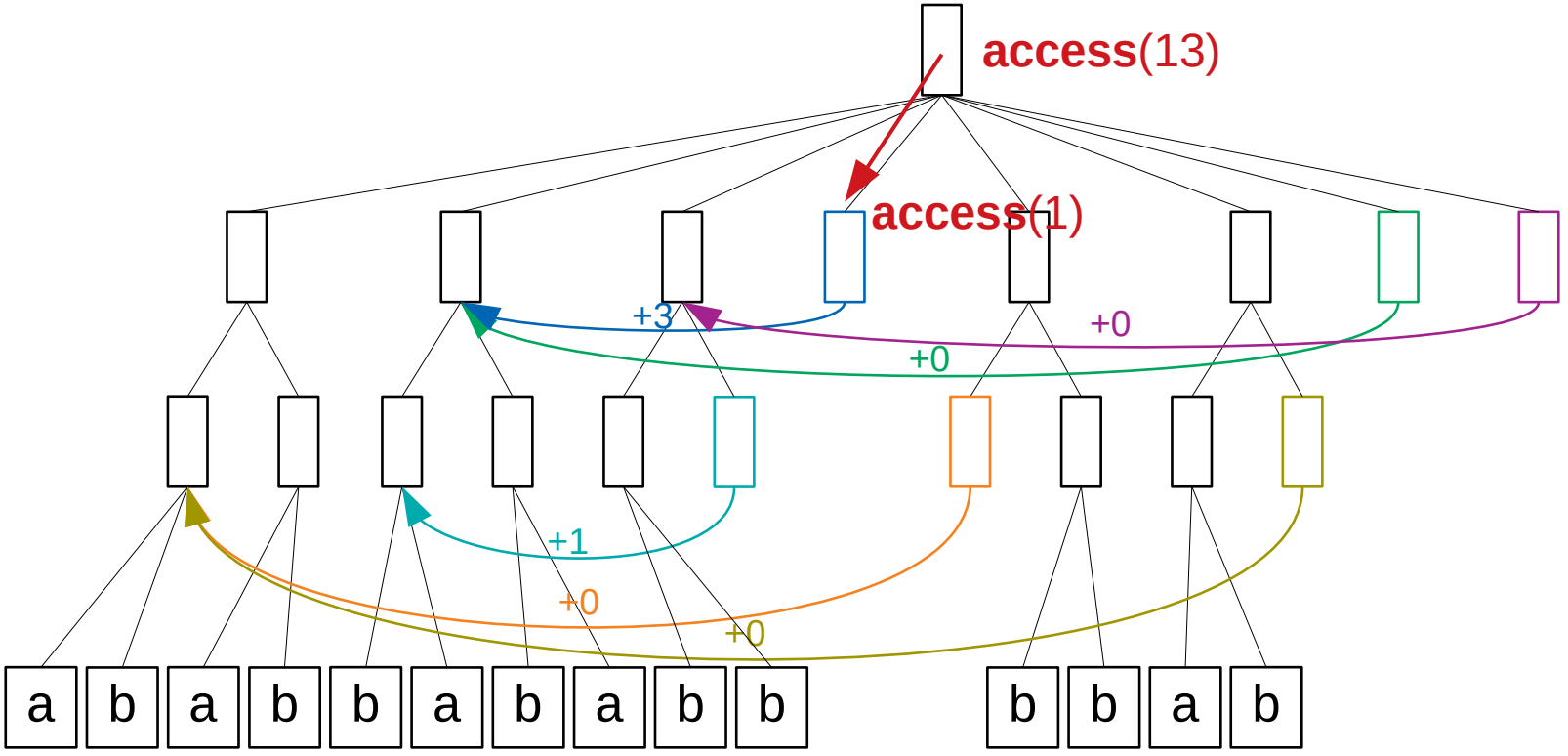

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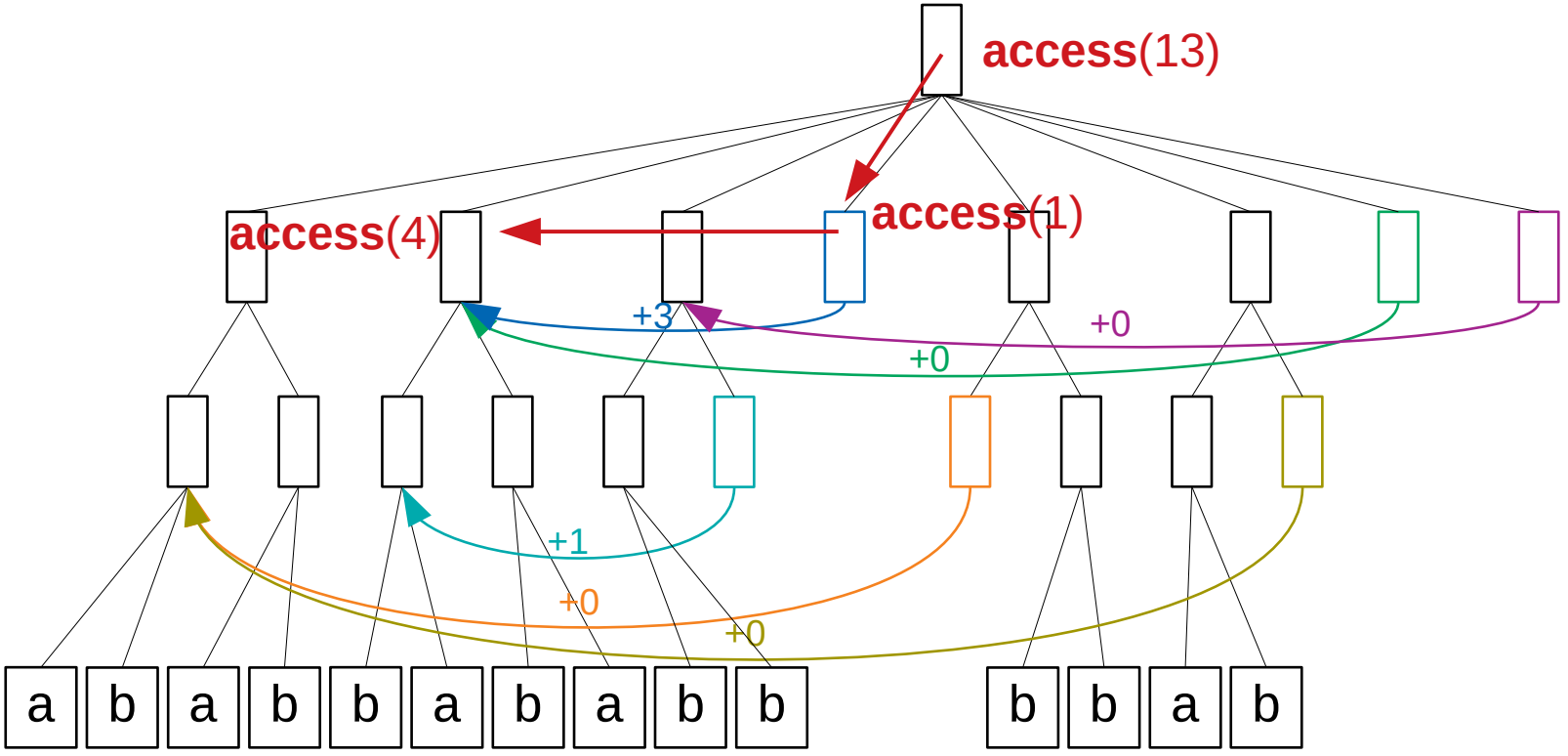

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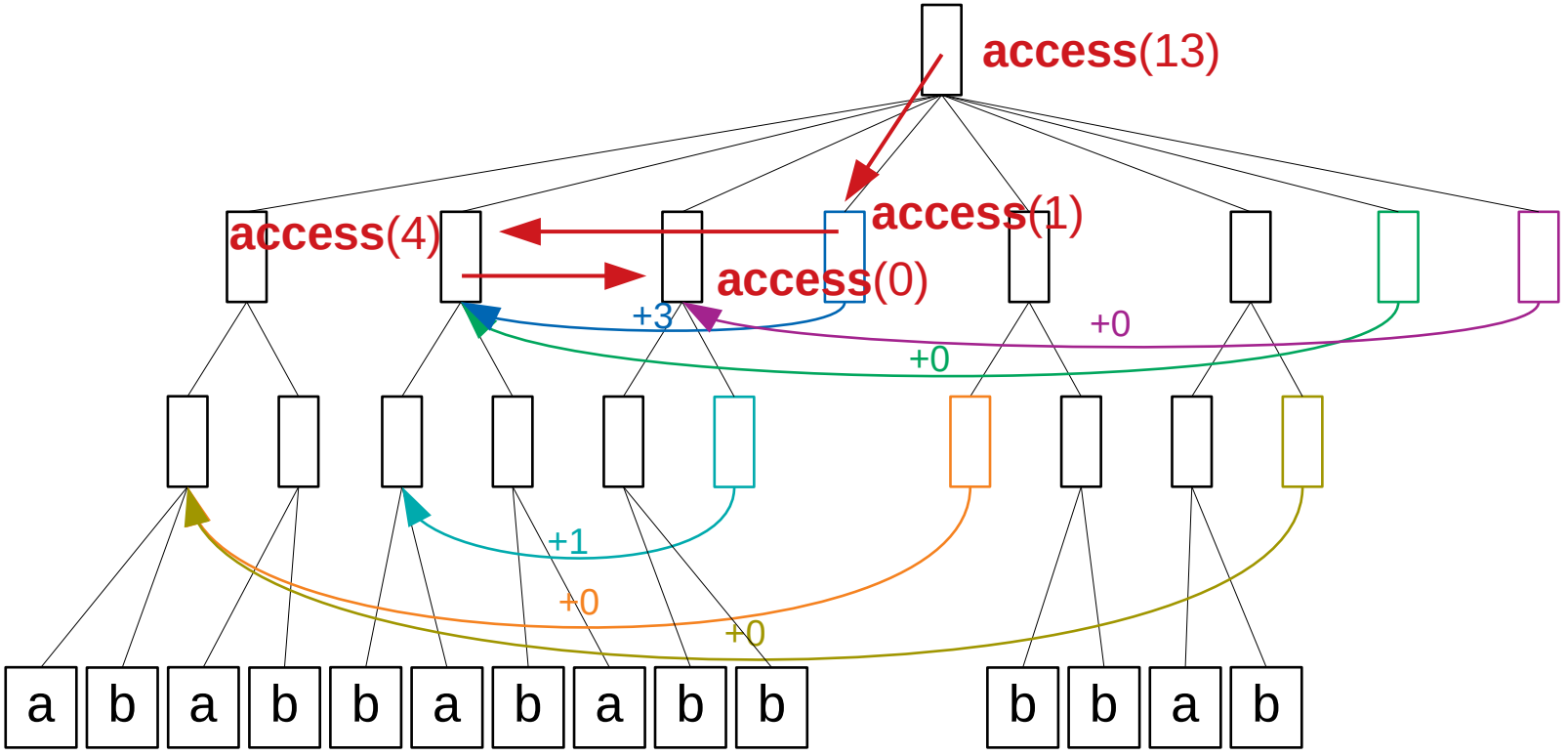

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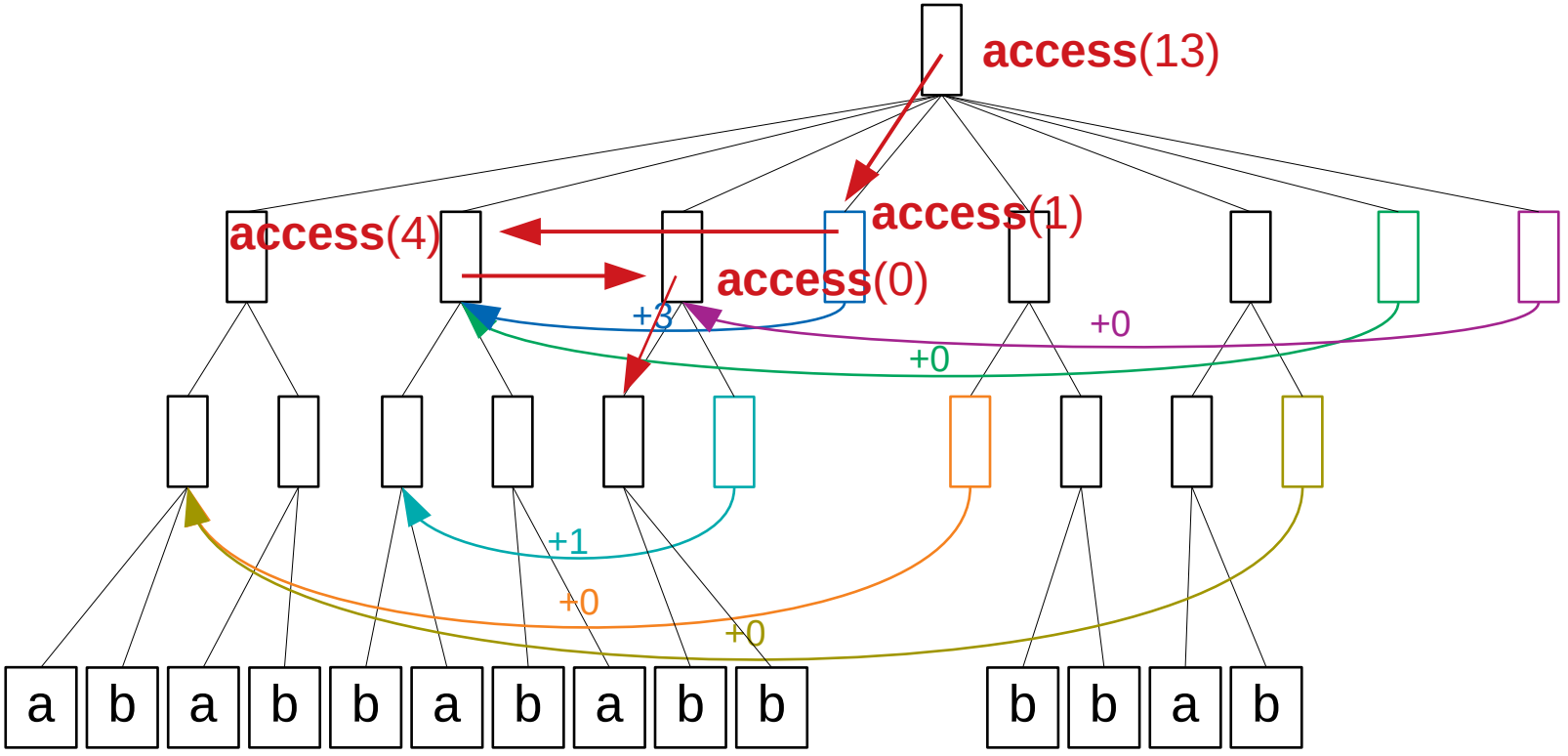

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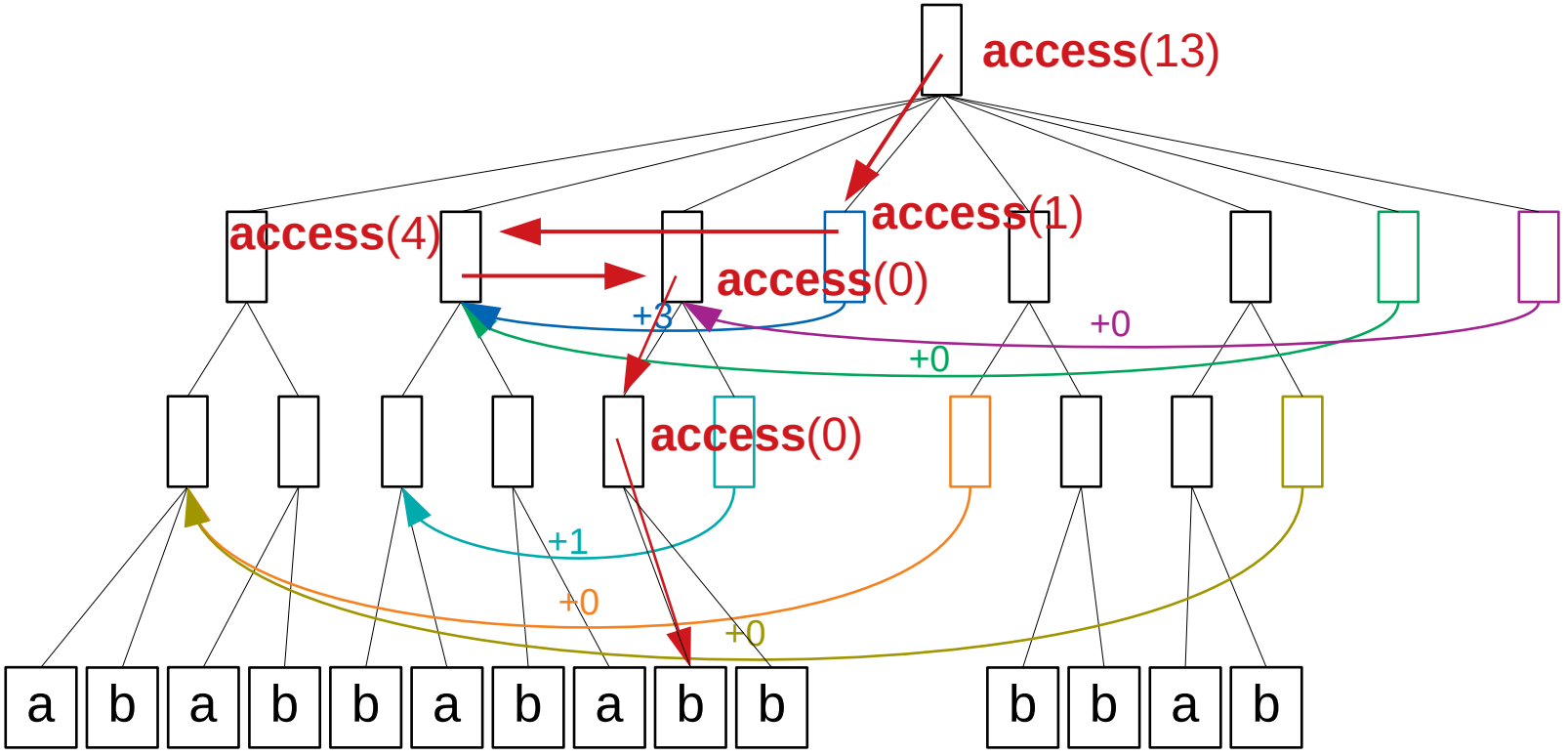

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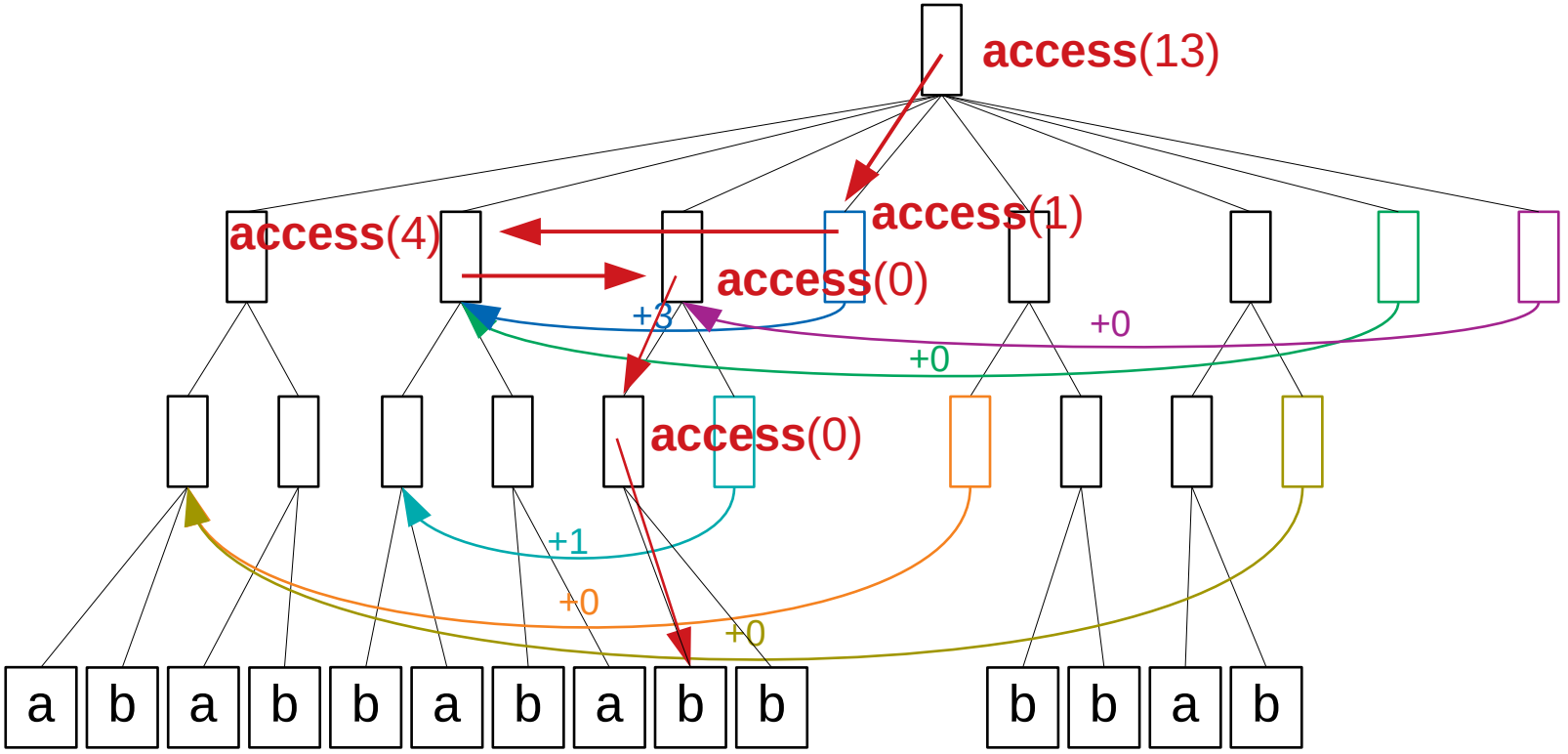

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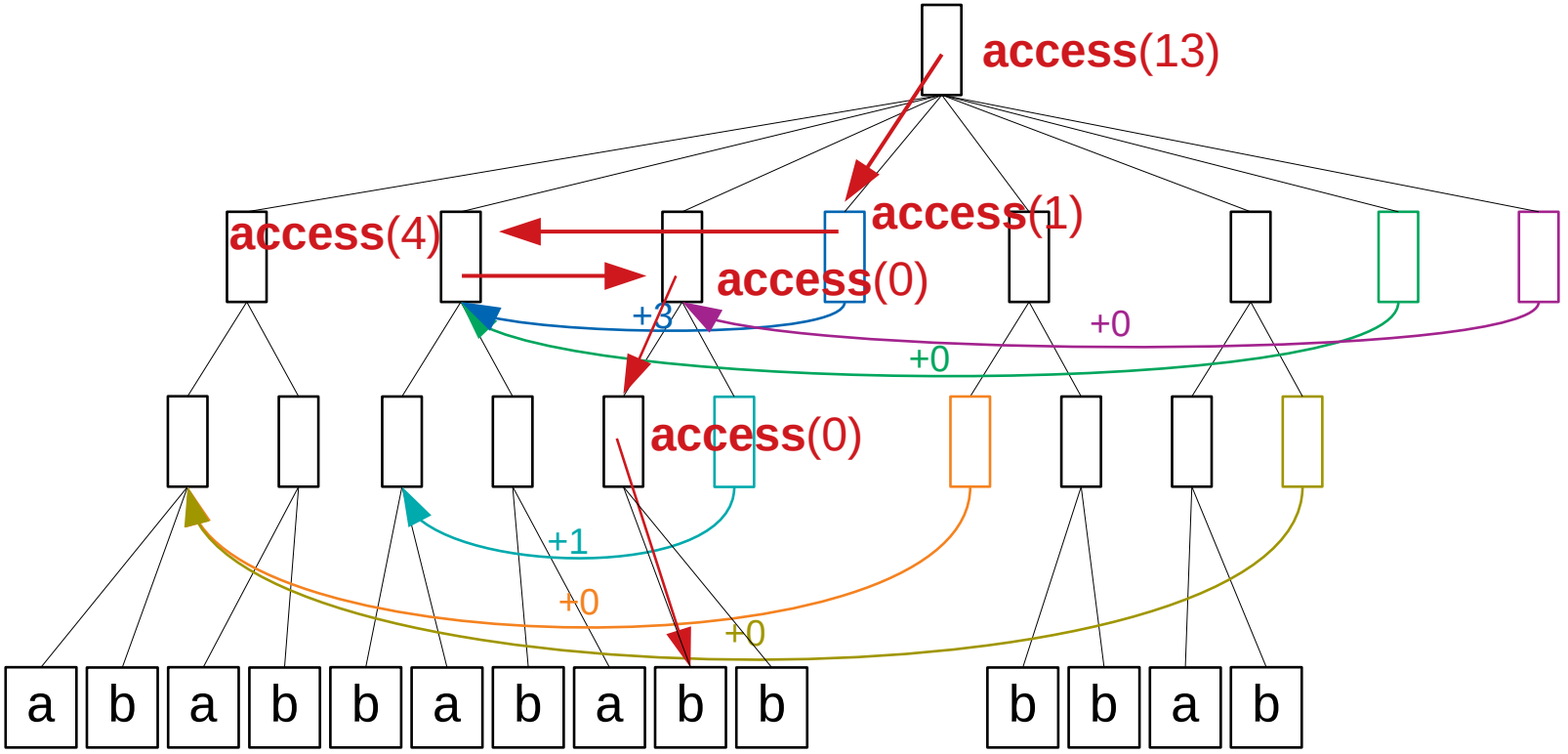

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- Extracting a substring of length m then requires time $O((1+m/\log_\sigma n)h)$

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  - [Kociumaka et al., LATIN 2020]